

SYMPLECTIC ANALOG OF CALABI'S CONJECTURE FOR CALABI–YAU THREEFOLDS

DMITRY V. EGOROV

ABSTRACT. In this paper we state an analog of Calabi's conjecture proved by Yau. The difference with the classical case is that we propose deformation of the complex structure, whereas the complex Monge–Ampère equation describes deformation of the Kähler (symplectic) structure.

1. CONJECTURE

Celebrated Calabi's conjecture [1] proved by Yau [2] gives a criterion for a Kähler manifold to have a vanishing Ricci curvature. It states that if the first Chern class of a manifold vanishes, then in a given Kähler class there exists a unique Ricci-flat metric. Yau proved conjecture by showing an existence of a solution for the complex Monge–Ampère equation.

In this paper we state an analog of Calabi's conjecture for threefolds. We consider the deformation of a holomorphic volume form, whereas the Kähler form is not deformed. We clarify our result of [3], where the main equation is non-scalar.

Conjecture 1.1. *Let M be a compact Kähler manifold of complex dimension $n = 3$ with $c_1(M) = 0$. By ω and $\Omega = \rho + \sqrt{-1}\sigma$ denote Kähler form and holomorphic volume form on M respectively. Consider equation*

$$(*) \quad (\Omega - \sqrt{-1}dd^s\varphi\Omega) \wedge (\bar{\Omega} + \sqrt{-1}dd^s\varphi\bar{\Omega}) = e^F \Omega \wedge \bar{\Omega},$$

or equivalently

$$(\rho + dd^s\varphi\sigma) \wedge (\sigma - dd^s\varphi\rho) = e^F \rho \wedge \sigma,$$

where function $F : M \rightarrow \mathbb{R}$ is normalized so that $\int_M e^F \omega^n = \int_M \omega^n$.

Then equation $(*)$ has unique (modulo adding a constant) solution $\varphi : M \rightarrow \mathbb{R}$ such that

- (1) $\tilde{\rho} = \rho + dd^s\varphi\sigma$ is positive;
- (2) $\tilde{\sigma} = \sigma - dd^s\varphi\rho$ is negative;
- (3) $\tilde{\rho}$ is dual to $\tilde{\sigma}$.

We state notions of positivity and duality for 3-forms and definition for d^s operator later in the paper. Now let us restate Conjecture 1.1 in a form free of PDE.

Conjecture 1.2. *Let M be a compact Kähler manifold of complex dimension $n = 3$ with $c_1(M) = 0$; let ω and Ω be a Kähler and holomorphic volume forms respectively. Then there exists a unique $\tilde{\Omega} \sim \Omega$ such that the Kähler metric $g_{ij} = \omega_{ik}\tilde{J}_j^k$ is Ricci-flat. Here \tilde{J} is the complex structure determined by $\tilde{\Omega}$.*

2010 *Mathematics Subject Classification.* Primary 32Q25, 35J96.

Key words and phrases. Kähler potential, Calabi–Yau manifold.

This work is supported in part by the Council of the Russian Federation President Grants (projects NSh-544.2012.1 and MK-842.2011.1).

The equivalence of Conjectures 1.1 and 1.2 follows from the following definitions of stable forms. The notion of a stable 3-form was introduced by Hitchin in [4, 5].

Definition 1.3. A real 3-form ρ on real 6-space V is called *stable* if the orbit of ρ under the natural action of $GL(V)$ is open in $\Lambda^3 V^*$.

Any stable form canonically determines a complex structure on V .

Definition 1.4. A real 3-form ρ on real symplectic 6-space is called *positive* if

- (1) ρ is stable;
- (2) ρ is primitive;
- (3) $\omega(X, J_\rho X) \geq 0$ for any $X \in V$, where ω is a symplectic form and J_ρ is the complex structure determined by ρ .

Any positive (negative) closed 3-form is a real (imaginary) part of the holomorphic volume form.

Definition 1.5. A stable 3-form ρ is called *dual* to a stable 3-form σ if

$$\sigma = J_\rho \rho.$$

Since we do not deform the Kähler structure, a symplectic differential operator is used.

Definition 1.6.

$$d^s \alpha := (-1)^{k+1} *_s d *_s \alpha,$$

where α is a k -form and $*_s$ is the symplectic Hodge star. The definition of $*_s$ is obtained by substituting the Riemannian metric for symplectic form in the definition of the usual Hodge star.

Remark 1.7. Note that d^s decreases degree of the form by one and anticommutes with d . There is also the dd^s -lemma analogous to the dd^c -lemma. Both lemmata hold on a Kähler manifold. See for example [6] and references therein.

Proposition 1.8. *The equation (*) differs from the complex Monge–Ampère equation.*

Proof. Suppose U is an open set of M with coordinate chart $\{x^i\}$ such that $\omega = dx^{12} + dx^{34} + dx^{56}$ on U ; then the global equation (*) is locally equivalent to the following equation on U

$$\begin{aligned} & (\varphi_{22} + \varphi_{33} + \varphi_{55})(\varphi_{11} + \varphi_{44} + \varphi_{66}) + (\varphi_{11} + \varphi_{44} + \varphi_{55})(\varphi_{22} + \varphi_{33} + \varphi_{66}) \\ & + (\varphi_{11} + \varphi_{33} + \varphi_{66})(\varphi_{22} + \varphi_{44} + \varphi_{55}) + (\varphi_{22} + \varphi_{44} + \varphi_{66})(\varphi_{11} + \varphi_{33} + \varphi_{55}) \\ & - (\varphi_{12} + \varphi_{34} + \varphi_{56})^2 - (-\varphi_{12} - \varphi_{34} + \varphi_{56})^2 - (\varphi_{12} - \varphi_{34} - \varphi_{56})^2 \\ & - (-\varphi_{12} + \varphi_{34} - \varphi_{56})^2 - 2[(\varphi_{13} - \varphi_{24})^2 + (\varphi_{36} + \varphi_{45})^2 + (\varphi_{15} - \varphi_{26})^2 \\ & + (\varphi_{16} + \varphi_{25})^2 + (\varphi_{35} - \varphi_{46})^2 + (\varphi_{14} + \varphi_{23})^2] = 1, \end{aligned}$$

where $\varphi_{ij} = \partial^2 \varphi / \partial x^i \partial x^j$.

Obviously, this equation differs from the complex Monge–Ampère equation. In fact, the operator of the equation coincides with a level- p Monge–Ampère operator considered in [7]. \square

Remark 1.9. The uniqueness of the $n = 3$ case is that a single stable real 3-form determines complex structure [4, 5]. In higher dimensions one needs a pair of real n -forms to determine a complex structure and therefore a pair of functions.

Remark 1.10. In [8] the generalized Monge–Ampère equation is defined in the sense of generalized complex structures. However, for the case of the Kähler geometry it reduces to the usual complex Monge–Ampère equation.

2. MIRROR KÄHLER POTENTIAL

In paper [3] we proposed a new equation for the 3-dimensional Calabi–Yau metrics and proved the solution existence theorem.

Theorem 2.1. *Let (M, ω) be a compact Kähler 3-manifold such that $c_1(M) = 0$; let $\Omega = \rho + i\sigma$ be a holomorphic volume form on M . Then the following equation*

$$(\rho + dd^s \alpha) \wedge (\sigma + dd^s \beta) = e^F \rho \wedge \sigma$$

or equivalently

$$(\Omega + dd^s \psi) \wedge (\bar{\Omega} + dd^s \bar{\psi}) = e^F \Omega \wedge \bar{\Omega}, \quad \psi = \alpha + \sqrt{-1}\beta$$

has solution $\alpha, \beta \in \Omega^3(M)$ such that

- (1) $\rho + dd^s \alpha$ and $\sigma + dd^s \beta$ are primitive stable 3-forms;
- (2) $\rho + dd^s \alpha$ is dual to $\sigma + dd^s \beta$ in the sense of the stable forms;
- (3) real function F is normalized: $\int_M e^F \rho \wedge \sigma = \int_M \rho \wedge \sigma$.

However, the main equation of [3] is non-scalar. We relate Theorem 2.1 with Conjecture 1.1 by conjecturing that there exists a *mirror Kähler potential*.

Conjecture 2.2. *Let M be a compact Calabi–Yau 3-fold with Kähler form ω and holomorphic volume form $\Omega = \rho + \sqrt{-1}\sigma$. Then for any $\tilde{\Omega} \sim \Omega$ there exists a function $\varphi : M \rightarrow \mathbb{R}$ such that*

$$\tilde{\Omega} = \Omega + \sqrt{-1}dd^s \varphi \Omega.$$

Example 2.3. Suppose \mathbb{C}^3 with a flat Hermitian metric $\sum dz^i \otimes d\bar{z}^i$; the Kähler form $\omega = (\sqrt{-1}/2) \sum dz^i \wedge d\bar{z}^i$ and holomorphic volume form $\Omega = dz^1 \wedge dz^2 \wedge dz^3$. Then the following identities hold:

$$\omega = dd^c \varphi, \quad \Omega = -(\sqrt{-1}/3)dd^s \varphi \Omega,$$

where $\varphi = \sum |z^i|^2$.

It would be interesting to establish a relationship between the classical and mirror Kähler potentials.

The author would like to thank D. Alexeevsky, G. Cavalcanti and M. Verbitsky for useful remarks during the conference "Geometric structures on complex manifolds – 2011" in Moscow. The author is grateful to I. A. Taimanov for support and useful remarks.

REFERENCES

1. E. Calabi, *On Kähler manifolds with vanishing canonical class*, Algebr. Geom. Topol. A symposium in honor of S. Lefschetz, pp. 78–89, Princeton University Press, Princeton, N.J., 1957.
2. S.-T. Yau, *On the Ricci curvature of compact Kähler manifold and the complex Monge–Ampère equation I*, Commun. pure appl. math. **31** (1978), 339–411.
3. D. Egorov, *New equation on the low-dimensional Calabi–Yau metrics*, Sib. Math. J. **52** (2011), no. 4, 651–654.
4. N.J.Hitchin, *The geometry of three-forms in six dimensions*, J. Diff. Geom. **55** (2000), 547–576.

5. N.J.Hitchin, *Stable forms and special metrics*, "Global Differential Geometry: The Mathematical Legacy of Alfred Gray", M. Fernández and J. A. Wolf (eds.), Contemp. Math. **288**, AMS, Providence (2001).
6. Li-Sheng Tseng and S.-T. Yau, *Cohomology and Hodge theory on symplectic manifolds: I*, arXiv:0909.5418 (2009).
7. F. Reese Harvey and H. Blaine Lawson Jr. *Foundations of p -convexity and p -plurisubharmonicity in Riemannian geometry*, arXiv:1111.3895 (2011).
8. Chris M. Hull, Ulf Lindström, Martin Roček, Rikard von Unge and Maxim Zabzine, *Generalized Calabi–Yau metric and generalized Monge–Ampere equation*, J. High Energy Phys. **2010** (2010), no. 8, 1–23.

AMMOSOV NORTHEASTERN FEDERAL UNIVERSITY, KULAKOVSKOGO STR. 48, 677000, YAKUTSK, RUSSIA

E-mail address: egorov.dima@gmail.com